Gas-Particle Flow in Convergent Nozzles at High Loading Ratios

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Nozzle flows of gas-particle mixtures are analyzed for particle loadings for which the particle volume may not be negligible. Nozzle shapes are such that the particle velocity is a constant fraction of the gas velocity. The influence of various parameters which enter into the analysis is discussed. For loading ratios greater than about five, the gas and particle temperatures remain substantially constant for the entire flow, and the assumption of isothermal flow reduces the flow equations to algebraic relationships. Throat conditions then can be obtained directly without having to compute the entire flow. A procedure is outlined to evaluate the ranges of the loading ratio for which the assumption of either isothermal flow or of negligible particle volume keeps errors below a desired level, and it is found that these ranges may overlap. An important intermediate range of the loading ratio therefore exists, typically from about five to fifty, for which both simplifying assumptions are permissible with a resultant extreme simplification of the relationships.

Nomenclature

a = speed of sound of gas
A = cross-sectional area of nozzle
c_p = specific heat of gas at constant pressure
c = specific heat of particles
C_D = drag coefficient
D = diameter of particles
F = thrust
g = acceleration of gravity
I_{sp} = specific impulse
K = velocity ratio, U_P/U_G
k = thermal conductivity of gas
m = mass flow rate of gas
Nu = Nusselt number
p = pressure
P = dimensionless pressure, p/p_r
R = individual gas constant
Re = Reynolds number, $\rho_G D(u_G - u_P)/\mu$
T = temperature
u = velocity
U = dimensionless velocity, $u/a_r = \zeta u/\alpha_r$
x = distance coordinate
X = dimensionless distance coordinate, $18\mu x/a_r\rho_P D^2$
α = equilibrium speed of sound in the gas-particle mixture
β = density ratio in the reservoir, $\rho_{G,r}/\rho_P$
γ = ratio of the specific heats of the gas
Γ = ratio of the specific heats for equilibrium in the gas-particle mixture, Eq. (20)
δ = specific heat ratio, c/c_p
ϵ = volume fraction of the particles
ζ = ratio of sound speeds, α_r/a_r
η = loading ratio
θ = dimensionless temperature, T/T_r
μ = viscosity of gas
ρ = density

Subscripts and superscripts

G = gas
P = particles
r = equilibrium reservoir
0 = nozzle inlet
$*$ = throat

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I. Introduction

NOZZLE flows of suspensions of small solid particles in a gas are of great technical importance. Many analyses have been published which deal with different aspects of such flows as indicated in reviews of the field.[1,2] For prescribed nozzle configurations, the flow equations have been solved numerically.[3-6] Variations of the nozzle shape led to the establishment of the conditions for optimum nozzle performance.[7,8] General insight into the behavior of the flow variables was obtained by assuming that the deviations from equilibrium flow are small enough that the equations may be linearized.[9-11]

If suitable assumptions are made for the variations of the particle velocity along the nozzle, the analytical work can be greatly simplified.[12-17] In particular, the nozzle equations become formally identical with those for a perfect gas with modified thermodynamic properties if the ratio of the particle velocity to the gas velocity is constant (constant-fractional-lag nozzle).[13,14] In this inverse approach, the shape of the nozzle is implied in the assumption for the particle velocity and must be determined subsequently by an appropriate integration. While the assumption of a constant-fractional-lag nozzle is arbitrary, it leads to nozzle shapes that are similar to those of actually used nozzles, at least in the vicinity of the nozzle throat, and experimental observations[14,18] yield reasonable agreement with experimental data. In any case, if the purpose of the analysis is to gain insight into the behavior of the flow variables rather than to obtain specific numerical results, this approach is no more arbitrary than one based on arbitrarily selected nozzle shapes.

Since the density of the particle material is usually about three orders of magnitude larger than that of the gas, the volume fraction occupied by the particles can be neglected as long as the particle loading is not too high. This condition is often satisfied, and all the analyses mentioned in the foregoing are explicitly or implicitly based on the assumption of a negligible particle volume. If the particle volume can be neglected, an equilibrium mixture of a perfect gas with solid particles behaves like a perfect gas with modified thermodynamic properties, but this simplification does not apply if the particle volume becomes significant. Since the particles may be treated as incompressible in comparison with the gas, the particle volume does not participate in a volume change of the mixture, and the particle volume fraction becomes an additional flow variable. Heavily loaded nozzle flows have been employed to inject a metallic fuel powder into a rocket combustion chamber.[19] Such flows also are of interest for certain nuclear reactors, for the conveying of particulate materials, in the chemical industry (fluidized beds[20]) and for other engineering applications.

The effect of a finite particle volume on the thermodynamic properties of gas-particle mixtures and on shock waves was investigated previously.[21,22] The present study extends this work to steady, quasi-one-dimensional nozzle flow. As in the case of negligible particle volume, an inverse method of assuming the particle velocity leads to considerable simplifications, but the equations still must be solved numerically. Such calculations indicate that both the gas and the particle temperatures remain substantially constant if the particle loading is high enough. A simplified theory based on isothermal flow then reduces the relationships to algebraic equations.

The limits of validity for the assumption of isothermal flow can be established by comparing the results with those obtained by numerical integration of the complete system of equations. This comparison reveals the existence of an intermediate range of the particle loading that is both high enough to treat the flow as isothermal and low enough to neglect the volume of the particles. With the simplifications that result from both of these assumptions, the solution of the flow equations can be expressed in an extremely simple form. The assumption of constant temperature for heavily loaded flows was postulated previously by Stockel.[20] Some of his relationships are similar to those presented here, but since he neglected the momentum contribution of the gas phase, his analysis is not valid at intermediate particle loadings for which the additional assumption of negligible particle volume applies.

II. Equations for Steady Nozzle Flow

The analysis will be based on the following usual assumptions, except that the particle volume will not be neglected. 1) The gas obeys the perfect-gas law and its specific heats are constant. 2) The particles are spherical, uniform in size, and incompressible. Their specific heat is constant, and the temperature is uniform within each particle. The limits of validity of this assumption are discussed elsewhere.[23] 3) The particles are uniformly distributed over the cross section of the nozzle, and their spacing is small compared with the duct dimensions. 4) The flow is one-dimensional. Boundary-layer effects and heat exchange with the walls are not considered. 5) The gas and particle velocity at any cross section represent average values. A detailed study by Panton[24] shows that proper averaging introduces some correction factors into the equations if the velocity profiles are not flat. Such refinements, which would have to be based on assumed velocity profiles, do not seem warranted at this time. 6) The effect of the particles on the gas flow appears initially in the wake of the particles and is then distributed over the entire cross section of the nozzle by mixing. In view of assumption 3, this mixing involves only a small volume and is assumed to take place instantaneously. 7) Random motion of the particles does not contribute to the pressure of the mixture. 8) No mass transfer takes place between the gas and the particles. 9) The drag coefficient and Nusselt number must be prescribed as functions of the particle Reynolds number. There exists considerable uncertainty about the appropriate relationships in the presence of many particles even for negligible volume of the particles,[1,2,25] and additional effects of the particle volume at high loading ratios may also become significant.[2,26] Most of the present analysis will involve only the dimensionless group ReC_D/Nu which is equal to 12 for Stokes drag ($C_D = 24/Re$) and $Nu = 2$. For single particles, the value of this group remains practically constant to much higher Reynolds numbers than those for which either Stokes drag or a constant Nusselt number apply.[27] The validity of this observation will be assumed for the present study.

Let ϵ denote the volume fraction of the gas-particle mixture occupied by the particles. The constant flow rate of the gas in a nozzle is then given by

$$m = (1 - \epsilon)\rho_G u_G A \tag{1}$$

The corresponding equation for the flow rate of the particles is

$$\eta m = \epsilon \rho_P u_P A \tag{2}$$

The constant ratio of the flow rate of the particles to that of the gas is denoted by η and is usually called the loading ratio.

The momentum equation for the entire mixture, which relates the pressure gradient dp/dx to the rate of change of the momentum flux per unit area of the gas and of the particles, may be written as

$$(1 - \epsilon)\rho_G u_G\, du_G/dx + \epsilon\rho_P u_P\, du_P/dx + dp/dx = 0 \tag{3}$$

The energy equation for the entire mixture indicates that the sum of the kinetic-energy flux and the enthalpy flux for the gas and the particles remains constant. As shown previously,[21] the enthalpy of the particle phase is given by $cT_P + p/\rho_P$. The energy equation in differential form therefore becomes[21,28]

$$u_G du_G + c_p dT_G + \eta(u_P du_P + c dT_P + dp/\rho_P) = 0 \tag{4}$$

The term $\eta dp/\rho_P$ is of order ϵ relative to the other terms, according to Eq. (2), and does not appear in analyses of nozzle flow with negligible particle volume.

The equation of motion of a particle of diameter D is derived in the Appendix and may be written as

$$(1 - \epsilon)u_P(du_P/dx) = C_D \tfrac{3}{4}(\rho_G/\rho_P)(u_G - u_P)^2/D \tag{5}$$

The rate of heat exchange between the gas and the particles depends on the heat-transfer coefficient which can be expressed in terms of a Nusselt number by kNu/D. The heat balance is then given by

$$u_P\, dT_P/dx = (6kNu/c\rho_P D^2)(T_G - T_P) \tag{6}$$

The last equation needed is the equation of state for a perfect gas

$$p = \rho_G R T_G \tag{7}$$

Equations (1–7) completely determine the flow if the inlet conditions and shape of the nozzle are prescribed. The condition $\epsilon = 0$, which corresponds to setting ρ_P equal to infinity [except in Eqs. (5) and (6)] leads to equations identical with those used in previous investigations.

It will be convenient to introduce dimensionless variables by using as reference state the conditions in an equilibrium reservoir in which the mass of the particles equals η times that of the gas. The equilibrium speed of sound in this reservoir can be expressed as[21]

$$\alpha_r^2 = \frac{1 + \eta\delta}{(1 + \eta)(1 + \gamma\eta\delta)(1 - \epsilon_r)^2}\, a_r^2 = \zeta^2 a_r^2 \tag{8}$$

With $\beta = \rho_{G,r}/\rho_P$, the particle volume fraction in the reservoir is given by

$$\epsilon_r = \beta\eta/(1 + \beta\eta) \tag{9}$$

As indicated earlier, the present study is based on an inverse procedure, that is, the gas velocity will be considered as the independent variable, and all other variables will be related to it. The method used is an extension of Kliegel's analysis for constant-fractional-lag nozzles[13,14] to flows with a finite particle volume. Accordingly, the nozzle is assumed

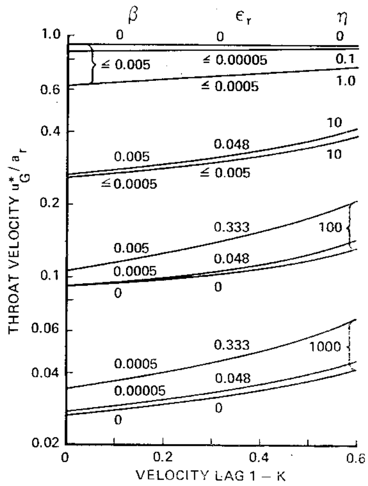


Fig. 1 Throat velocity in constant-fractional-lag nozzles for $\gamma = 1.4$ and $\delta = 1$.

to have a shape for which

$$u_P/u_G = U_P/U_G = K \tag{10}$$

where K is a constant. The quantity $1 - K$ thus indicates the velocity lag of the particles, and $K = 1$ corresponds to equilibrium flow.

Division of Eq. (6) by Eq. (5) eliminates the position coordinate, and combination with Eq. (10) and introduction of the particle Reynolds number leads to

$$d\theta_P/dU_G = [(1 - \epsilon)K/\delta(1 - K)](\theta_G - \theta_P)/U_G \tag{11}$$

where $ReC_D/Nu = 12$ (assumption No. 9) and a Prandtl number $\mu c_P/k = \frac{2}{3}$ have been assumed.

If Eq. (3) in differential form is combined with Eqs. (1, 7, 8, and 10) and expressed in dimensionless variables, one obtains an equation which may be arranged to

$$dP/dU_G = -\gamma(1 + \eta K)(1 - \epsilon)PU_G/\theta_G \tag{12}$$

Similarly, Eq. (4) yields

$$\frac{d\theta_G}{aU_G} = -\left[\delta\eta\frac{d\theta_P}{dU_G} + \frac{(\gamma - 1)\beta\eta}{\gamma}\frac{dP}{dU_G} + (\gamma - 1)(1 + \eta K^2)U_G\right] \tag{13}$$

From the derivatives of Eqs. (1) and (2), one obtains with the aid of Eqs. (7) and (10)

$$\frac{dA}{dU_G} = -A\left[\frac{1}{U_G} + (1 - \epsilon)\left(\frac{1}{P}\frac{dP}{dU_G} - \frac{1}{\theta_G}\frac{d\theta_G}{dU_G}\right)\right] \tag{14}$$

and

$$d\epsilon/dU_G = -\epsilon[(1/U_G) + (1/A)(dA/dU_G)] \tag{15}$$

It is interesting to note that in Kliegel's analysis ($\epsilon = 0$), the constant velocity ratio K implies that the derivative $d\theta_P/d\theta_G$ is also constant. In the present analysis, this derivative may be formed from the preceding equations, and it is not difficult to show that it is constant only if the particle volume is neglected.

Equations (11–15) form a complete system that can be solved numerically for the variables θ_P, P, θ_G, A, and ϵ. Such calculations must start from prescribed conditions at the nozzle inlet which may be arbitrarily selected. One may assume that the flow originates from the reference reservoir, but the assumption of equilibrium in the reservoir is incompatible with the velocity ratio according to Eq. (10) unless $K = 1$. A transition duct between the reservoir and the nozzle inlet is therefore implied in which the desired velocity ratio is established. The inlet area is arbitrarily designated as $A_0 = 1$. If the selected value of $U_{G,0}$ is sufficiently small, the pressure and temperature may be set equal to the reservoir value or $P_0 = \theta_{G,0} = \theta_{P,0} = 1$. The particle velocity is given by $U_{P,0} = KU_{G,0}$. For high loading ratios, the equilibrium speed of sound in the reservoir may be much smaller than the speed of sound in the gas phase alone. All calculations are therefore based on $U_{G,0} = 0.1\zeta$, that is, the inlet gas velocity is taken as 10% of the equilibrium speed of sound in the mixture. The inlet value for the particle volume fraction follows from Eqs. (1, 2, 7, and 10) as

$$\epsilon_0 = \beta\eta P_0/(K\theta_{G,0} + \beta\eta P_0) \tag{16}$$

Integration of the system of Eqs. (12–15), with the selected starting conditions was based on a Runge-Kutta method for simultaneous first-order differential equations[29] and programed for an IBM 7044 computer. It was assumed that the nozzle discharged into a region of sufficiently low pressure that the flow would be choked, and the calculations proceeded until the throat of the nozzle was reached. The throat, where the derivative dA/dU_G vanishes, was found by interpolation between the two consecutive integration steps for which this derivative changes sign. Integration was not extended beyond the throat. Such results would be questionable because the particles tend to remain concentrated near the nozzle axis,[3,30] and the assumption of uniform distribution of the particles over the cross section of the nozzle would then not be satisfied. Indeed, a comparison of computed and observed flows by Stockel[20] yielded good agreement only in the region upstream of the nozzle throat.

Results obtained in the described manner relate the flow variables at any point in the nozzle to the gas velocity at that point. The shape of the nozzle required to produce this flow involves integration of Eq. (5) for a specified drag coefficient. For the purpose of finding the effect of neglecting the particle volume, it appears satisfactory to assume Stokes drag ($C_D = 24/Re$). In dimensionless coordinates, Eq. (5) then becomes

$$dX/dU_G = [K^2/(1 - K)](1 - \epsilon) \tag{17}$$

which can be integrated together with the previous equations. It is seen that for $\epsilon = 0$, the gas velocity, and therefore also the particle velocity, become linear functions of the distance travelled, a solution that is well known from previous investigations.[2,13–16]

III. Results of the Numerical Integration

The most important parameters of the analysis are the loading ratio η, the velocity ratio K and the density ratio in the reservoir β, but the specific-heat ratios γ and δ also enter into the calculations. Computations, based on the relationships given in the preceding section, were performed for loading ratios between zero and 1000 and velocity ratios between 0.4 and 1.0. The density ratio β was varied between zero, which corresponds to neglecting the particle volume, and 5×10^{-3}; typically, it is near 5×10^{-4}.

Calculations for all possible combinations for several values of each parameter would have been prohibitive, but the influence of each parameter was explored while keeping the others fixed at typical values. The effect of varying δ between 0.1 and 10 and γ between 1.1 and 1.67 is found to be insignificant at a loading ratio of 100. The results presented in the following are based on $\gamma = 1.4$ and $\delta = 1.0$. At the same loading ratio, a decrease of β from 5×10^{-4} to zero results in an increase of the pressure by about 2%, while an increase to 5×10^{-3} leads to a decrease of the pressure by about 12%. The corresponding effect on the temperature is negligible.

The influence of β on the velocity is important as shown in Fig. 1, where the dimensionless throat velocity u_G^*/a_r is plotted as a function of the velocity lag $1 - K$ for various combinations of the loading ratio η and density ratio β. It is seen that the velocity decreases markedly with increasing loading ratio and, for a given loading ratio, increases considerably with increasing velocity lag, particularly at higher values of η. The reservoir volume fraction of the particles is determined by Eq. (9) and is indicated for each curve. For any value of η, the particle volume fraction significantly affects the velocity only when it approaches 5%. For lower values of ϵ_r, its effect is too small to be shown in the figure.

In contrast to the wide variations of the gas velocity, the throat pressure P^* is rather insensitive to the choice of the parameters and always lies between about 0.5 and 0.6. In some cases, this variation is equal to the effect of neglecting the particle volume. The relationship between particle volume and pressure is further discussed in the next section.

Integration of the simultaneous equations given in the preceding section yields both the cross-sectional area and the distance along the nozzle axis as function of the gas velocity. The nozzle shape may therefore be obtained from this parametric representation. The results for $\eta = 100$ and $K = 0.6$ are shown in Fig. 2 where the ratio of the nozzle diameter to the throat diameter is plotted as a function of the dimensionless distance from the throat for various values of ϵ_r. As in the case of the velocity, the effect of the particle volume becomes noticeable when it approaches 5%. For $\epsilon_r = 0$, the solution becomes identical with Kliegel's solution.[14]

The specific thrust and impulse of such nozzles may be determined from the computed data. For a discharge into a vacuum, the specific thrust may be expressed in dimensionless variables by

$$F/A^*p_r = \gamma(1 + \eta K)(1 - \epsilon^*)P^*U_G^{*2}/\theta_G^* + P^* \quad (18)$$

The corresponding specific impulse, defined as the ratio of the thrust to the total weight flow, is given by

$$\frac{gI_{sp}}{a_r} = \frac{U_G^*}{1 + \eta}\left[1 + \eta K + \frac{\theta_G^*}{\gamma(1 - \epsilon^*)U_G^{*2}}\right] \quad (19)$$

Values of the dimensionless specific impulse are related in Fig. 3 to the velocity lag for several values of the loading ratio and for β between zero and 5×10^{-4}. The corresponding values of ϵ_r are indicated in the figure. The impulse decreases drastically with increasing loading ratio and also decreases with increasing velocity lag. It is again found that the particle volume fraction leads to noticeable effects when it approaches 5%.

The relationship between specific impulse and loading ratio is shown in Fig. 4 for $K = 0.6$ and $\beta = 5 \times 10^{-4}$. Also shown in the figure is the corresponding specific thrust which goes through a minimum and then increases with increasing loading ratio. This behavior can be qualitatively explained by noting that the decrease of the discharge velocity with increasing loading ratio is accompanied by a simultaneous increase of the total mass flow. These effects combine to produce thrust variations of not more than about 5% over the entire range of the loading ratio, while the dimensionless specific impulse decreases from about 1.6 for $\eta = 0$ to about 0.04 for $\eta = 1000$. The effect of the particle volume is too small to be shown for the specific impulse. For the thrust, it

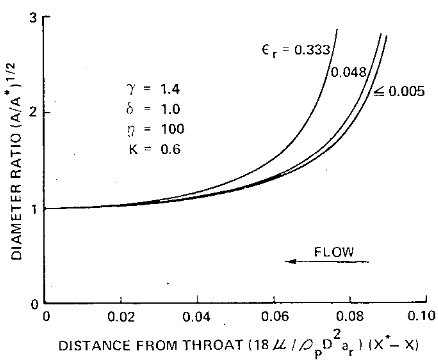
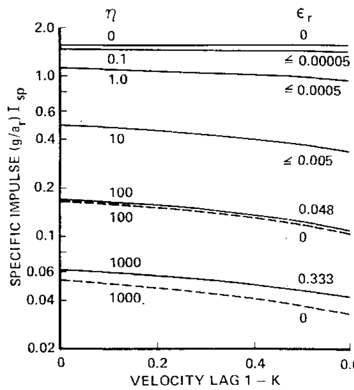


Fig. 2 Effect of particle volume on the shape of a constant-fractional-lag nozzle.

Fig. 3 Specific impulse of constant-fractional-lag nozzles for $\gamma = 1.4$ and $\delta = 1$. The solid lines are for $\beta = 5 \times 10^{-4}$ and the broken lines for $\beta = 0$.



amounts to a few percent at the highest loading ratios, as indicated by the broken line.

In Fig. 5, the throat temperatures are plotted as functions of the loading ratio for several values of the velocity ratio. For equilibrium flow, the gas and particle temperatures must, of course, be equal. The most conspicuous feature of the numerical results is the small variation of the gas and particle temperatures for larger loading ratios. It can be seen that the temperature drop between the reservoir and the nozzle throat is smaller than 2% if the loading ratio exceeds ten. Consequently, an approximate analysis of the flow may be based on the assumption that both the gas and the particle temperatures are equal and constant throughout the entire flow. The consequences of this assumption are discussed in the following section.

IV. Analysis of Isothermal Flow

The small temperature drop at high loading ratios, indicated by Fig. 5, can be explained by considering that the heat capacity of the particle phase is so large compared with that of the gas, that the cooling of the gas during the nozzle expansion is compensated by heat transfer from the particles without a significant lowering of the particle temperature. Stockel[20] used this qualitative reasoning to justify his assumption of isothermal flow, but its limits of validity can be established only by more precise results, such as those given in Fig. 5. Stockel also neglected the contribution of the gas to the momentum of the mixture, that is, the first term on the left-hand side of Eq. (3); this additional assumption will not be made here. The assumption of isothermal flow may also be justified by considering the effective ratio of the specific heats for an equilibrium gas-particle mixture, which is given by

$$\Gamma = \gamma(1 + \eta\delta)/(1 + \gamma\eta\delta) \quad (20)$$

It is easily demonstrated that, with increasing loading ratio, the ratio of the specific heats rapidly approaches unity, the value appropriate for isothermal flow.

The assumption of isothermal flow eliminates the two variables T_G and T_P since $T_G = T_P = T_r$, or $\theta_G = \theta_P = 1$. As a

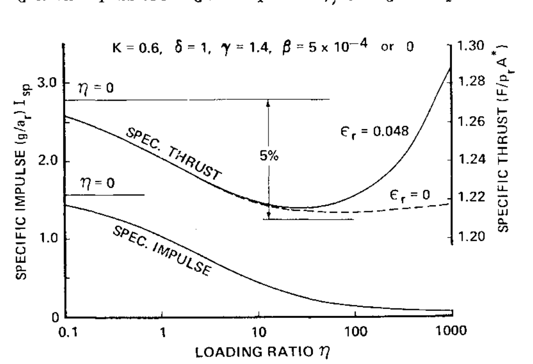


Fig. 4 Specific thrust and impulse of a constant-fractional-lag nozzle.

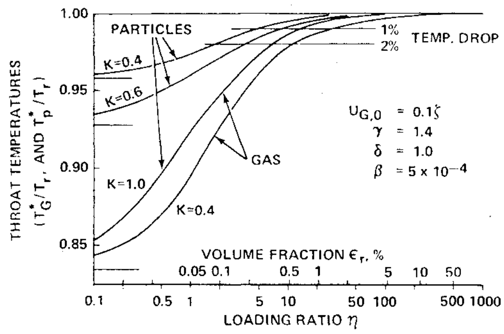


Fig. 5 Gas and particle temperatures at the throat of constant-fractional-lag nozzles.

consequence, Eqs. (4) and (11) are no longer required. Note that it is not permissible to omit the temperature terms in Eq. (4) and use the equation formed by the remaining terms, since the assumption of isothermal flow is a consequence of the kinetic energy terms being small compared with the enthalpy terms.

For isothermal flow, Eqs. (1, 2, 7, and 10) yield a relationship between the particle volume fraction and the pressure which may be written in the form of a conservation equation

$$(1 - \epsilon)P/\epsilon = (1 - \epsilon_0)P_0/\epsilon_0 = K/\beta\eta \qquad (21)$$

If Eqs. (1) and (10) are substituted into the differential form of Eq. (3), ρ_G is replaced by p/RT_r from Eq. (7), and ϵ is eliminated by using Eq. (21), one obtains, in dimensionless form,

$$\gamma(1 + \eta K)U_G dU_G + (1 + \beta\eta P/K)dP/P = 0 \qquad (22)$$

Integration of this equation yields

$$U_G^2 = U_{G,0}^2 + \frac{2}{\gamma(1 + \eta K)}\left[\ln\frac{P_0}{P} + \frac{\epsilon_0}{1 - \epsilon_0}\left(1 - \frac{P}{P_0}\right)\right] \qquad (23)$$

where $\beta\eta/K$ has been expressed in terms of ϵ_0 and P_0 from Eq. (21). The integration constant is determined by the prescribed variables $U_{G,0}$, P_0, and ϵ_0. Combination of Eqs. (1) and (7) yields

$$(1 - \epsilon)PU_G A = \text{const} \qquad (24)$$

and, if $(1 - \epsilon)$ is eliminated with the help of Eq. (21), leads to

$$A/A_0 = \epsilon_0 U_{G,0}/\epsilon U_G \qquad (25)$$

The entire nozzle flow can therefore be computed if the inlet conditions are prescribed. It is most convenient to select the pressure as the independent variable, because only simple algebraic relationships then have to be evaluated.

Conditions at the nozzle throat also can be readily obtained. The logarithmic derivatives of Eqs. (23) and (24) with dA/A = 0 yield, after combination with Eq. (21), the condition

$$dU_G/U_G^* + [K/(K + \beta\eta P^*)]dP/P^* = 0 \qquad (26)$$

which is valid only at the throat, as indicated by asterisks. Division of Eq. (22), which is valid everywhere, by Eq. (26) leads to relationships between the throat velocity and the throat pressure or the particle volume fraction

$$\gamma(1 + \eta K)U_G^{*2} = \left(1 + \frac{\epsilon_0}{1 - \epsilon_0}\frac{P^*}{P_0}\right)^2 = \frac{1}{(1 - \epsilon^*)^2} \qquad (27)$$

where $\beta\eta/K$ has been eliminated by means of Eq. (21). If this equation is substituted into Eq. (23), one obtains a transcendental equation for the throat pressure

$$2\ln\frac{P^*}{P_0} + \left(\frac{\epsilon_0}{1 - \epsilon_0}\frac{P^*}{P_0}\right)^2 + 4\frac{\epsilon_0}{1 - \epsilon_0}\frac{P^*}{P_0} = \gamma(1 + \eta K)U_{G,0}^2 - 1 + \frac{2\epsilon_0}{1 - \epsilon_0} \qquad (28)$$

which can be solved numerically for prescribed values of P_0, $U_{G,0}$, and ϵ_0. The other throat variables then follow from Eqs. (21) and (27). Throat conditions are therefore obtainable directly without integration of the flow equations along the entire nozzle. The nozzle shape would have to be determined by integration of Eq. (5) as before, or if Stokes drag is assumed, by integration of Eq. (17).

Inspection of the preceding equations reveals that considerable further simplifications would result if the particle volume fraction could be neglected ($\beta = \epsilon = 0$). In particular, the throat conditions would then be described by

$$2\ln(P^*/P_0) = \gamma(1 + \eta K)U_{G,0}^2 - 1 \qquad (29)$$

$$U_G^* = 1/\gamma(1 + \eta K) \qquad (30)$$

and

$$A^*/A_0 = U_{G,0}P_0/U_G^*P^* \qquad (31)$$

It is especially noteworthy that the value of U_G^* is independent of the inlet conditions of the nozzle. If the inlet velocity is low enough that the first term on the right-hand side of Eq. (29) can be neglected, then P_0 becomes approximately equal to unity, and the throat pressure ratio becomes simply $P^* = e^{-1/2} = 0.6065$.

It remains to establish the range of validity of the isothermal-flow solution and to see whether or not there exists a significant range of the loading ratio for which both isothermal flow and a negligible particle volume fraction may be assumed. This question may be answered by comparing results based on these assumptions with "exact" data obtained by numerical integration of the complete equations. Such a comparison is shown in Fig. 6 for the gas velocity, pressure, and cross-sectional area at the throat as a function of the loading ratio for $K = 0.6$ and $\beta = 5 \times 10^{-4}$. The corresponding values of the particle volume fraction under reservoir conditions are also indicated. The solid curves are based on the equations which contain the ϵ-terms, and it can be seen that practically perfect agreement with the "exact" data is approached at higher loading ratios. If the ϵ-terms are neglected, the results deviate from "exact" data as indicated by the broken lines. Nevertheless, there exists a range of the loading ratio, roughly from 5 to 50, for which the errors are smaller than about 3%. The errors increase for higher loading ratios because the effects of the particle volume fraction should not be neglected, and for lower loading ratios because the assumption of isothermal flow becomes unsatisfactory.

For a desired accuracy of the results, the lower limit of the loading ratio for which the assumption of isothermal flow is permissible may be estimated from the relationship between pressure and temperature for an equilibrium (isentropic) flow[21]

$$P^* = \theta^{*\Gamma/(\Gamma-1)} = e^{-1/2} \qquad (32)$$

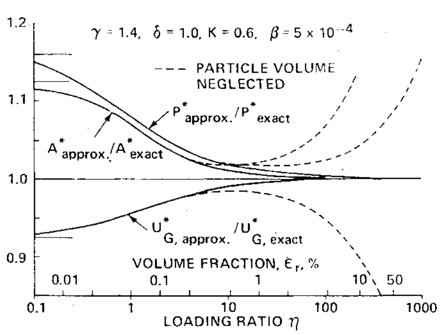


Fig. 6 Comparison of approximate results derived for isothermal flow and exact results obtained by integration of the complete equations.

For small temperature changes, this equation yields

$$\Gamma/(\Gamma - 1) = 0.3935/(1 - \theta^*) \tag{33}$$

For any prescribed temperature change $1 - \theta^*$ which is to be considered negligible, the maximum value of Γ then follows, and Eq. (20) yields the corresponding lower limit of the loading ratio. The upper limit, above which the particle volume should not be neglected, can be obtained from the equations derived for isothermal flow.

V. Conclusions

The presented analysis is based on nozzles for which the particle velocity is a constant fraction of the gas velocity along the entire nozzle. The shape of the nozzle is implied in this assumption and must be determined together with the other flow variables. Numerical solutions of the flow equations are used to assess the influence of the various parameters which enter into the analysis. It is found that the specific heats of the gas and of the particles have only an insignificant effect on the results. Increase of the particle loading causes a large reduction of the exhaust velocity from the nozzle which decreases with increasing velocity lag of the particles. In contrast, pressure and temperature are affected to a much lesser extent. Allowing for the particle volume influences the results only when it exceeds about 5%.

Of particular interest is the observation that the gas and particle temperatures remain practically constant along the entire nozzle if the particle loading is high enough, that is, if the heat content of the particles is so large that cooling of the gas during nozzle expansion can be compensated by heat transfer from the particles without a significant temperature change. If the assumption of isothermal flow is made, the differential flow equations can be reduced to algebraic equations.

This theory can be further simplified if the particle volume is neglected, and numerical results reveal the unexpected existence of an appreciable intermediate range of the particle loading for which both isothermal flow and negligible particle volume may be assumed. The loading ratio for which these conditions hold depends on the properties of the mixture and the allowable errors. Typically, it lies between about 5 and 50 for errors of about 3%. The errors become larger at lower loading ratios because the flow is no longer isothermal and at higher loading ratios because the effect of the particle volume becomes significant. A procedure is outlined which may be used for specific constituents of the mixture to determine the range of the loading ratio for which the errors remain below a specific level.

The errors which result from the assumption of isothermal flow do not become very large even for small loading ratios. For the example shown in Fig. 6, the maximum error of the throat velocity approaches only 7.5% as the loading ratio approaches zero, while it reaches 16% for the throat pressure and 12.5% for the throat area. The corresponding error of the gas temperature is 16% while that of the particle temperature is only about 7%, as indicated by Fig. 5. In view of the extreme simplicity of the equations, an analysis based on isothermal flow and negligible particle volume may therefore be useful even for low particle loading if high accuracy is not needed.

The specific thrust of a nozzle which discharges a gas-particle mixture into a low-pressure region varies by only a few percent over the entire range of the loading ratio, because the reduction of the exit velocity caused by the particles is approximately compensated by the increase of the total mass flow. In contrast, the specific impulse decreases drastically with increasing particle loading. These findings are not greatly affected by consideration of the particle volume.

In general, the throat conditions for a nonequilibrium flow can be obtained only by computing the entire flow from the nozzle inlet to the throat, because the maximum flow rate that can pass through a given throat cannot be prescribed beforehand. Some iteration, involving either the flow rate or the throat area, is then needed to compute the throat conditions. The possibility of obtaining the throat conditions directly without having to compute the entire flow is thus a distinct advantage of the presented approximate theory. This convenient feature is a consequence of the constant velocity ratio but requires, in addition, that either the particle volume can be neglected, or that the flow can be considered as isothermal.

Appendix: Equation of Motion for a Particle in a Gas-Particle Mixture

A single particle moving under the influence of the surrounding gas is affected by viscous drag, the pressure gradient in the gas, and forces that result from the continuous adjustment of the gas flow around the particle. Usually, the density of the gas is so small compared with that of the particle that only the viscous drag is important.[31,32] If many particles are present, one must distinguish between terms in the equations that are of the order of the density ratio ρ_G/ρ_P and those of the order of the particle volume fraction ϵ. These two quantities are not independent of each other but are related by $\epsilon = (\eta\rho_G/\rho_P)/(K + \eta\rho_G/\rho_P)$ according to Eqs. (1), (2), and (10). While terms of order ρ_G/ρ_P, or ϵ/η, can be neglected for the same reasons as in the case of single particles, terms of order ϵ may become important at sufficiently high loading ratios. Panton[24] discussed also possible contributions of statistical fluctuations of the particle motion including a Reynolds stress which, in the absence of further knowledge, will not be considered. (Experimental determinations of drag coefficients for shock-tube flows have been qualitatively explained by assuming fluctuations in the particle motion.[25])

In view of the foregoing, the equation of motion for a particle in a gas-particle mixture can be written as

$$\frac{\pi D^3}{6}\rho_P u_P \frac{du_P}{dx} = C_D \frac{1}{2}\rho_G(u_G - u_P)^2 \frac{\pi D^2}{4} - \frac{\pi D^3}{6}\frac{dp}{dx} \tag{A1}$$

and substitution for the pressure gradient from Eq. (3) yields

$$u_P \frac{du_P}{dx} = C_D \frac{3}{4}\frac{\rho_G}{\rho_P}\frac{(u_G - u_P)^2}{D} + \epsilon u_P \frac{du_P}{dx} \tag{A2}$$

where terms of the order ρ_G/ρ_P have been neglected as before.† The first term represents the dominating viscous drag, while the contribution of the pressure gradient is of the nature of a relatively small correction term. For small loading ratios, the particle volume fraction is negligible, and the entire second term may be omitted. Equation (5) then results if the second term is combined with the term on the left-hand side of the equation. Since terms of order ρ_G/ρ_P have been neglected, this result becomes doubtful under unusual conditions such as extremely high pressures (large ρ_G) or hollow particles (small ρ_P).

References

[1] Hoglund, R. F., "Recent Advances in Gas-Particle Nozzle Flows," *ARS Journal*, Vol. 32, No. 5, May 1962, pp. 662–671.

[2] Soo, S. L., *Fluid Dynamics of Multiphase Systems*, Blaisdell, Waltham, Mass., 1967.

[3] Bailey, W. S. et al., "Gas Particle Flow in an Axisymmetric Nozzle," *ARS Journal*, Vol. 31, No. 6, June 1961, pp. 793–798.

[4] Glauz, R. D., "Combined Subsonic-Supersonic Gas-Particle Flow," *ARS Journal*, Vol. 32, No. 5, May 1962, pp. 773–775.

[5] Hultberg, J. A. and Soo, S. L., "Two-Phase Flow Through

† After completion of this manuscript, the author's attention was drawn to a brief note[33] where a relationship is derived that is equivalent to Eq. (A2).

a Nozzle," *Astronautica Acta*, Vol. 11, No. 3, May-June, 1965, pp. 207–216.

[6] Hoffman, J. D. and Lorenc, S. A., "A Parametric Study of Gas-Particle Flows in Conical Nozzles," *AIAA Journal*, Vol. 3, No. 1, Jan. 1965, pp. 103–106.

[7] Marble, F. E., "Nozzle Contours for Minimum Particle-Lag Loss," *AIAA Journal*, Vol. 1, No. 12, Dec. 1963, pp. 2793–2801.

[8] Hoffman, J. D. and Thompson, H. D., "Optimum Thrust-Nozzle Contours for Gas-Particle Flows," *AIAA Journal*, Vol. 5, No. 10, Oct. 1967, pp. 1886–1887.

[9] Rannie, W. D., "Perturbation Analysis of One-Dimensional Heterogeneous Flow in Rocket Nozzles," *ARS Progress in Astronautics and Rocketry: Detonation and Two-Phase Flow*, Vol. 6, edited by S. S. Penner and R. A. Williams, Academic Press New York, 1962, pp. 117–144.

[10] Marble, F. E., "Dynamics of a Gas Containing Small Solid Particles," *Combustion and Propulsion (5th AGARDograph Colliquium)*, Pergamon, Oxford, 1963, pp. 175–213.

[11] Marble, F. E., "The Role of Approximate Analytical Results in the Study of Two-Phase Flow in Nozzles," Publication 188, Aug. 1967, Guggenheim Jet Propulsion Center, California Institute of Technology.

[12] Gilbert, M., Davis, L., and Altman, D., "Velocity Lag of Particles in Linearly Accelerated Combustion Gases," *Jet Propulsion*, Vol. 25, No. 1, Jan. 1955, pp. 26–30.

[13] Kliegel, J. R., "One Dimensional Flow of a Gas-Particle System," Paper 60-5, Jan. 1960, IAS.

[14] Kliegel, J. R., "Gas Particle Nozzle Flow," *Ninth Symposium (International) on Combustion*, Academic Press, New York, 1963, pp. 811–826.

[15] Soo, S. L., "Gas Dynamic Processes Involving Suspended Solids," *AIChE Journal*, Vol. 7, No. 3, Sept. 1961, pp. 384–391.

[16] Duban, P. and Nicolas, J., "The Effect of Solid Particles on the Flow in a Nozzle," *La Recherche Aéronautique*, No. 92, Jan.-Feb. 1963, pp. 17–29.

[17] Hassan, H. A., "Exact Solutions of Gas-Particle Nozzle Flows," *AIAA Journal*, Vol. 2, No. 2, Feb. 1964, pp. 395–397.

[18] Brown, B., "Rocket Motor Performance—Nozzle Configuration Effects in Two-Phase Flow," *Pyrodynamics*, Vol. 3, No. 3–4, Oct. 1966, pp. 221–233.

[19] Burge, H. L., Roberts, R. W., and Zettle, E. V., "Dense Phase Transport of Powdered Metals for a Tripropellant Rocket System," *Chemical Engineering Progress Symposium Series*, Vol. 60, No. 52, 1964, pp. 30–38.

[20] Stockel, I. H., "High-Speed Flow of Fluidized Solids in Changing Area," *Chemical Engineering Progress Symposium Series*, Vol. 58, No. 38, 1962, pp. 106–120.

[21] Rudinger, G., "Some Effects of Finite Particle Volume on the Dynamics of Gas-Particle Mixtures," *AIAA Journal*, Vol. 3, No. 7, July 1965, pp. 1217–1222.

[22] Schmitt-v. Schubert, B., "Strömungen von Gasen mit Festen Teilchen," thesis, 1968, Technische Hochschule Darmstadt.

[23] Rudinger, G., "Relaxation in Gas-Particle Flow," *Non-equilibrium Flows. Vol. 1 of Gasdynamics*, edited by P. P. Wegener, Dekker, New York, 1969, pp. 119–161.

[24] Panton, R., "Flow Properties for the Continuum Viewpoint of a Non-Equilibrium Gas-Particle Mixture," *Journal of Fluid Mechanics*, Vol. 31, Pt. 2, Jan. 1968, pp. 273–303.

[25] Rudinger, G., "Effective Drag Coefficients for Gas-Particle Flow in Shock Tubes," *Transactions of the ASME, Ser. D: Journal of Basic Engineering*, Vol. 92, No. 1, March 1970, pp. 165–172.

[26] Ergun, S., "Fluid Flow through Packed Columns," *Chemical Engineering Progress*, Vol. 48, No. 2, Feb. 1952, pp. 89–94.

[27] Carrier, G. F., "Shock Waves in a Dustry Gas," *Journal of Fluid Mechanics*, Vol. 4, Pt. 4, Aug. 1958, pp. 376–382.

[28] Kraiko, A. N. and Sternin, L. E., "Theory of Flows of a Two-Velocity Continuous Medium Containing Solid or Liquid Particles," *Journal of Applied Mathematics and Mechanisms (PMM)*, Vol. 29, Pt. 3, 1965, pp. 482–496.

[29] Fox, L., *Numerical Solution of Ordinary and Partial Differential Equations*, Addison-Wesley, Reading, Mass., 1962, p. 25.

[30] Gilbert, M., Allport, J., and Dunlap, R., "Dynamics of Two-Phase Flow in Rocket Nozzles," *ARS Journal*, Vol. 32, No. 12, Dec. 1962, pp. 1929–1930.

[31] Hinze, J. O., *Turbulence*, McGraw-Hill, New York, 1959, pp. 354–355.

[32] Fuchs, N. A., *The Mechanics of Aerosols*, Macmillan, New York, 1964, pp. 70–77.

[33] Schmitt-V. and Schubert, B., "Bemerkungen zu 'Analysis of Normal Shock Waves in a Gas-Particle Mixture' von T. D. Varma and N. K. Chopra, ZAMP 18, 650 (1967)," *Zeitschrift für Angewandte Mathematik und Physik*, Vol. 20, Pt. 2, March 1969, pp. 272–275.